49th IEEE Conference on Decision and Control
December 15-17, 2010
Hilton Atlanta Hotel, Atlanta, GA, USA

Numerical Solutions to Noisy Systems

Sanja Živanović and Pieter Collins

***Abstract*— A numerical method for rigorous over-approximation of a solution set of an input-affine system whose inputs represent some bounded noise is presented. The method gives high order error for a single time step and a uniform bound on the error over the finite time interval. The approach is based on the approximations of inputs by linear functions at each time step. We derive the single-step error in the one-dimensional case, and give the formula for the error in higher dimensions. As an illustration of the theory presented, a rigorous numerical result is given.**

I. INTRODUCTION

In this paper, we study systems of the form

$$\dot{x}(t) = f(x(t), v(t)), \;\; x(t) \in \mathbb{R}^n, \;\; v(t) \in V \subset \mathbb{R}^m,$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a smooth function, V is a compact set and $v(t)$ is a measurable function known as the disturbance input. The reason we assume that disturbance input is a measurable function is because we can rewrite the above equation in the form

$$\dot{x}(t) \in f(x(t), V) = F(x(t)),$$

where F is a multivalued map, and obtain a *differential inclusion* describing the evolution. Differential inclusions are a generalization of differential equations having multivalued right-hand sides [2], [5], [13]. In this paper, we are interested in algorithms for computing rigorous over-approximations to the reachable sets of a differential inclusion.

The correspondence between differential inclusions and systems with bounded noise has attracted people from control community to have an interest in differential inclusions and their numerical solution. For example, if the system is not completely controllable, one may want to know (compute) the set of all solutions. On the other hand, if we want to compute a solution of high dimensional system $\dot{x}(t) = f(x(t))$, by performing system reduction, and we would like to obtain an over-approximation of the solution set, then desired reduced system corresponds to a differential inclusion $\dot{z}(t) = g(z(t)) \pm \varepsilon$. In particular, when there is an absence of a control law, or there is a variety of available dynamics, or there is an uncertainty involved, one needs differential inclusions. In general, differential inclusions have applications in many areas of science, such as mechanics, electrical engineering, the theory of automatic control, economical, biological, and social macrosystems.

S. Živanović is with the Department of Scientific Computing and Control Theory, Centrum Wiskunde en Informatica, Amsterdam, 1098 XG, The Netherlands Sanja.Zivanovic@cwi.nl

P. Collins is with the Department of Scientific Computing and Control Theory, Centrum Wiskunde en Informatica, Amsterdam, 1098 XG, The Netherlands Pieter.Collins@cwi.nl

The first result on the computation of the solution set of a differential inclusion was given by Puri, Varaiya and Borkar [11], who considered Lipschitz differential inclusions, $\dot{x}(t) \in F(x)$, $x(0) = x_0$ and gave a polyhedral method for obtaining an approximation of the solution set $S(x_0)$ to an arbitrary known accuracy. In the case where F is only upper-semicontinuous with compact, convex values, it is possible to compute arbitrarily accurate over-approximations to the solution set, as shown in [4].

Some different techniques and various types of numerical methods have been proposed as approximations to the solution set of a differential inclusion. For example, ellipsoidal calculus was used by Valyi and Kurzhanski [10], Lohner-type algorithm by Zgliczynski and Kapela [16], grid methods by Puri, Varaiya, and Borkar [11], also by Beyn and Rieger [3], discrete approximations by Dontchev and Farkhi [6], also by Grammel [8]. However, these algorithms either do not give rigorous over-approximations, or are approximations of low-order (Euler approximations with a first-order single-step truncation error). Essentially, the only algorithms that could give higher order error estimates are the ones that use grids. However, higher order discretization of a state space highly effects efficiency of the algorithm. It was noted in [3] that if one is trying to obtain higher order error estimates on the solution set of differential inclusions then grid methods should be avoided.

In this paper we give a higher-order method for the rigorous over-approximation of the solution set of an input-affine differential inclusion. The method we give yields third-order single-step truncation error, though in principle it should be possible to give a higher-order method.

In Section II, we give some notes on mathematical background for the theory used and state the problem that we consider. In Section III, we give details on the approximation equation, and we give formulas for computation of the local error obtaining $O(h)$, $O(h^2)$, $O(h^3)$. In Section IV, we derive a formula for error computation in one-dimension. The algorithm is presented in Section V, and a rigorous numerical result is given in Section VI. We conclude the paper in Section VII, by giving an overview of the work presented, and pointing out possible extensions.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. Mathematical Background

Below we give several results on differential inclusions and their computability. For further inquiry on the theory of differential inclusions see [2], [5], [13], for computability

theory see [15], and for results on computability of differential inclusions see [11], [4].

First, by an over-approximation P' to a set P, we mean that $P \subseteq P'$. A solution of a differential inclusion is an absolutely continuous function $x : [0,T] \to \mathbb{R}^n$, such that for almost all $t \in [0,T]$, $x(\cdot)$ is differentiable at t and $\dot{x}(t) \in F(x(t))$. If $x(0) = x_0$, the solution set $S(x_0) \subset C([0,T],\mathbb{R}^n)$ is defined as

$$S(x_0) = \{x(\cdot) \in C([0,T],\mathbb{R}^n) \mid x(\cdot) \text{ is a solution of } \dot{x} \in F(x) \text{ with } x(0) = x_0\}.$$

The *solution set at time t*, $S(x_0,t) \subset \mathbb{R}^n$, is defined as

$$S(x_0,t) = \{x(t) \in \mathbb{R}^n \,|\, x(\cdot) \in S(x_0)\}.$$

Theorem 1: Let $D \subset \mathbb{R}^n$ and $F : [0,T] \times D \rightrightarrows \mathbb{R}^n$ be an upper semicontinuous set-valued mapping, with non-empty, compact and convex values. Assume that $||F(t,x))|| \leq c(1 + ||x||)$, for some constant c, is satisfied on $[0,T]$. Then for every $x_0 \in D$, there exists an absolutely continuous function $x : [0,T] \to \mathbb{R}^n$, such that $x(t_0) = x_0$ and $\dot{x}(t) \in F(t,x(t))$ for almost all $t \in [0,T]$.

The proof of the theorem can be found in [2] and [5]. We continue by stating the result on upper-semicomputability of differential inclusions. The proof of the following theorem can be found in [4].

Theorem 2: Let F be an upper-semicontinuous multivalued function with compact and convex values. Consider the initial value problem $\dot{x} \in F(x)$, $x(0) = x_0$, where F is defined on some open domain $V \subset \mathbb{R}^n$. Then the solution operator $x_0 \mapsto S(x_0)$ is upper-semicomputable in the following sense: Given an enumerator of all tuples $(I, K_1, ..., K_m)$ such that $F(\bar{I}) \subset \cup_{i=1}^m K_i$, it is possible to enumerate all tuples $(I, J, K_1, ..., K_m)$ where $I, K_1, ..., K_m$ are open rational boxes and J is an open rational interval such that for every $x_0 \in I$, every solution ξ with $\xi(0) = x_0$ satisfies $\xi(\bar{J}) \subset \cup_{i=1}^m K_i$.

In this paper we shall need the multidimensional mean value theorem, which can be found in standard textbooks on real analysis book, e.g., see [14]. We use the following form of the theorem.

Theorem 3: Let $V \subset \mathbb{R}^n$ be open, and suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable on V. If $x, x+h \in V$ and $L(x; x+h) \subseteq V$, i.e., line between x and $x+h$ belongs to V, there is a $t \in [0,1]$ such that

$$f(x+h) - f(x) = \int_0^1 Df(x+th)dt \cdot h$$

where Df denotes Jacobian matrix of f, and integration is understood component-wise.

In Section IV, we will use inequality given below to get certain estimates on the error. The inequality uses logarithmic norm which we define first. We take the following definition and the theorem from [9]. The proof of the theorem can also be found in there.

Definition 4: Let Q be a square matrix. Then

$$\lambda(Q) = \lim_{h \to 0^+} \frac{||I + hQ|| - 1}{h}$$

is the *logarithmic norm* of Q.

Theorem 5: Suppose that we have estimates $\lambda(Df(t,z(t))) \leq l(t)$, for $z(t) \in \text{conv}\{x(t), y(t)\}$ and $||\dot{y}(t) - f(t,y(t))|| \leq \delta(t)$, $||x(t_0) - y(t_0)|| \leq \rho$. Then for $t \geq t_0$ we have

$$||y(t) - x(t)|| \leq e^{\int_{t_0}^t l(s)ds} \left(\rho + \int_{t_0}^t e^{-\int_{t_0}^t l(s)ds} \delta(s) \right).$$

The numerical results given in Section VI were obtained by using the tool for reachability analysis and verification of real systems, Ariadne (see [1]). The main functionality used in Ariadne are *Taylor models* which were developed by Berz and Makino over the past decade. Taylor models (see [12] and references therein), provide rigorous functional inclusion method. The approximations are given in a form of polynomial (defined over a suitably small domain) plus an interval remainder.

Definition 6: Let $f : D \subset \mathbb{R}^v \to \mathbb{R}$ be a function that is $(n+1)$ times continuously partially differentiable on an open set containing the domain D. Let $x_0 \in D$ and P an n^{th} order polynomial of f around x_0. Let I be an interval such that $f(x) \in P(x - x_0) + I$, for all $x \in D$. Then we call the pair (P, I) an n^{th} order Taylor model of f around x_0 on D.

B. Problem Formulation

In this paper, we restrict attention to the input-affine system

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^m g_i(x(t))v_i(t); \quad x(t_0) = x_0. \tag{1}$$

We assume that

- $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^2 function,
- each $g_i : \mathbb{R}^n \to \mathbb{R}^n$ is a nonzero continuous function,
- $v_i(\cdot)$ is a measurable function such that $|v_i(t)| \leq A_i$ for some positive real numbers A_i.

The solution set $S(x_0,t)$ of the system (1) exists (on some interval of existence, say $[0,T]$), and is equivalent (e.g., see [2]) to the solution set of the differential inclusion obtained by replacing each $v_i(t)$ by the set $[-A_i, +A_i]$ in (1).

The aim of this paper is to give methods to compute concrete over-approximations S_k to the solution sets $S(x_0,t_k)$ for a sequence of rational time points t_k. We can also derive over-approximations to the solution set over a time step, $S(x_0, [t_k, t_{k+1}])$.

III. SINGLE-STEP APPROXIMATION

In order to compute over-approximation to the solution set in (1), we compute solution set of a different (an approximate) differential equation and add the uniform error bound on the difference of the two solutions. In this section, we set up the problem of an approximation and give formulas for the uniform error bound.

A. Setting up an approximation

Let $[0,T]$ be the interval of existence of (1), let h be a time step size, and set $t_k = hk$. For $x \in \mathbb{R}^n$ and $v(\cdot) \in L^\infty([t_k, t_{k+1}]; \mathbb{R}^m)$, define $\phi(x_k, v(\cdot))$ to be the point x_{k+1} which is the value at time t_{k+1} of the solution of $\dot{x}(t) =$

$f(x(t),v(t))$ with $x(t_k)=x_k$. For each time step we need to compute an over-approximation S_{k+1} to the set

$$\{\phi(x_k,v(\cdot)) \mid x_k \in S_k \text{ and } v(\cdot) \in L^\infty([t_k,t_{k+1}];\mathbb{R}^m)\}.$$

We assume that all disturbances are restricted to the space of bounded measurable functions which is infinite-dimensional. Thus, we aim to approximate the set of all solutions by restricting the disturbances to a finite-dimensional space. Consider a set of approximating functions $w(a,\cdot)$ parameterised by $a \in A \subset \mathbb{R}^p$. We then need to find an error bound ε such that

$$\forall v_k \in L^\infty([t_k,t_{k+1}];\mathbb{R}^m),\ \exists a_k \in A \quad \text{s.t. } ||\phi(x_k,v_k(\cdot)) - \phi(x_k,w(a_k,\cdot))|| \le \varepsilon.$$

Setting $\tilde{\phi}(x,a)=\phi(x_k,w(a,\cdot))$, i.e. $\tilde{\phi}$ also denotes the solution of $\dot{x}(t)=f(x_k,w(a_k,\cdot))$, with $x(t_k)=x_k$, at $t=t_{k+1}$, we obtain the approximation

$$S_{k+1}=\{\tilde{\phi}(x_k,a_k)\pm\varepsilon \mid x_k \in S_k \text{ and } a_k \in A\}.$$

Define the approximate system at time step k by

$$\dot{y}(t)=f(y(t))+\sum_{i=1}^m g_i(y(t))w_i(t); \quad y(t_k)=y_k,\ t\in[t_k,t_{k+1}]. \tag{2}$$

We would like to choose approximating functions $w_i = w(a_i,\cdot):[t_k,t_{k+1}]\to\mathbb{R}$, depending on $x(t_k)$ and $v_i(\cdot)$, such that the solution of (2) is an approximation of high order to the solution of (1).

B. Formulas for the uniform error bound

Our aim is to compute an over-approximation of the solution set in (1). In order to do that, first, we compute an over-approximation of the solution set at time t_1. Initially, we can assume $x(t_0)=y(t_0)$. We denote the over-approximation at time t_1 by S_1, which we take to be the set of initial points of original system and its approximation for the next time step. Therefore, we assume that $x(t_1)=y(t_1)$. By repeating this procedure at each time step, we can, without loss of generality, assume that $x(t_k)=y(t_k)$, for all $k\ge 0$.

If we take $w_i(\cdot)=0$, then an approximation of $O(h)$ is obtained. Using Theorem 5, one can get the formula for the calculation of the error. Let B be a bound on the solutions of (1) and (2) for all $t\in[0,T]$, and assume that

$$|v_i(\cdot)|\le A_i,\ ||f(z(t))||\le K,\ ||g_i(z(t))||\le K_i \quad ||Df(z(t))||\le L,\ ||D^2f(z(t))||\le H,\ \lambda(Df(\cdot))\le\Lambda, \tag{3}$$

for each $i=1,...,m$, and for all $t\in[0,T]$, and $z(\cdot)\in B$. The formula that gives a bound on the local error at t_{k+1} is

$$||x(t_{k+1})-y(t_{k+1})|| \le \frac{e^{\Lambda h}-1}{\Lambda}\left(\sum_{i=1}^m A_iK_i\right). \tag{4}$$

Note that $(e^{\Lambda h}-1)/\Lambda \approx h+\Lambda h^2/2+...$ is $O(h)$. In fact, inequality above was obtained in [16] as the logarithmic norm estimate on the perturbations of ODEs.

To derive an error of higher order we consider the following two cases.

1) Simple equations: In addition to the assumptions that we had so far, in this subsubsection we assume that $g_i(\cdot)$ are C^1 functions with $||Dg_i(\cdot)||\le L_i$. Let $w_i(t)$ be constant, in particular, suppose that we obtain (bounds on) $w_i(\cdot)$ by solving the following integral equality

$$\int_{t_k}^{t_{k+1}} v_i(t)-w_i(t)\,dt=0,$$

for all $i=1,...,m$. It is easy to see that $|w_i(t)|\le A_i$, for all $t\in[t_k,t_{k+1})$. Hence, the local error can be computed using the formula

$$||x(t_{k+1})-y(t_{k+1})|| \le h^2\left(\sum_{i=1}^m L_iA_i\right)\left(K+\sum_{i=1}^m A_iK_i\right) + 2h\left(L+\sum_{i=1}^m L_iA_i\right)\left(\sum_{i=1}^m A_iK_i\right)\frac{e^{\Lambda h}-1}{\Lambda}.$$

which is of $O(h^2)$. However, with the same assumptions above we can get an approximation of $O(h^2)+O(h^3)$. The formula is

$$\begin{aligned}&(1-Lh-h\sum_{i=1}^m L_iA_i)||x(t_{k+1})-y(t_{k+1})|| \le \\ &\frac{h^2}{2}\left(HK+\sum_{i=1}^m HK_iA_i+L^2+\left(3L+2\sum_{i=1}^m A_iL_i\right)\sum_{i=1}^m A_iL_i\right)\times\left(\sum_{i=1}^m K_iA_i\right)\frac{e^{\Lambda h}-1}{\Lambda} \\ &+h^2\left(\frac{L}{4}\left(\sum_{i=1}^m K_iA_i\right)+\left(\sum_{i=1}^m L_iA_i\right)\left(2K+3\sum_{i=1}^m K_iA_i\right)\right).\end{aligned} \tag{5}$$

Now, if we assume that $w_i(t)=a_{i,0}+a_{i,1}(t-(t_k+h/2))$ is linear, and

$$\begin{aligned}\int_{t_k}^{t_{k+1}} v_i(t)-w_i(t)\,dt&=0\\ \int_{t_k}^{t_{k+1}}(v_i(t)-w_i(t))(t-(t_k+h/2))\,dt&=0,\end{aligned} \tag{6}$$

hold (which we use to find bounds for $|a_{i,0}|$, $|a_{i,1}|$ and $|w_i(t)|$), then the formula is

$$\begin{aligned}&(1-Lh-h\sum_{i=1}^m L_iA_i)||x(t_{k+1})-y(t_{k+1})|| \le \\ &\frac{7h^2}{8}\left(HK+\frac{5}{2}\sum_{i=1}^m HK_iA_i+L^2+\left(\frac{9}{2}L+5\sum_{i=1}^m A_iL_i\right)\sum_{i=1}^m A_iL_i\right)\times\left(\sum_{i=1}^m K_iA_i\right)\frac{e^{\Lambda h}-1}{\Lambda} \\ &+\frac{h^2}{4}\left(\sum_{i=1}^m L_iA_i\right)\left(11K+\frac{69}{2}\sum_{i=1}^m K_iA_i\right) \\ &+\frac{7h^3}{48}\left(\sum_{i=1}^m HK_iA_i+LL_iA_i\right)\left(K+\sum_{i=1}^m K_iA_i\right)\end{aligned}$$

It is not clear which of the above formula gives tighter bounds. This will depend on the values of constants and the value of time step h. However, if the noise is additive or $m = 1$, then the last case (when $w_i(\cdot)$ is linear) can be derived even more to give us an error of $O(h^3)$.

2) Full approximation: In this paper, we aim to obtain an approximation error of $O(h^3)$ per time-step. To do this, we require that $w_i(t) = (w_i^1(t), ..., w_i^n(t))$ is a vector valued, whose each component $w_i^j(t) = a_{i,0}^j + a_{i,1}^j(t-(t_k+h/2))$ is a linear function and satisfies the following integral equalities

$$\begin{aligned}&\int_{t_k}^{t_{k+1}} v_i(t)g_i^j(x(t)) - w_i^j(t)g_i^j(y(t))\,dt = 0\\ &\int_{t_k}^{t_{k+1}} (v_i(t)g_i^j(x(t)) - w_i^j(t)g_i^j(y(t)))(t-(t_k+h/2))\,dt = 0.\end{aligned} \tag{7}$$

for all $i = 1, ..., m$ and $j = 1, ..., n$. Solving the above system of equations we can get bounds for coefficients $a_{i,0}^j$, $a_{i,1}^j$, and bounds on $w_i^j(t)$. If we denote by $0 < K_i^{j,l} \le |g_i^j(z(t))| \le K_i^{j,u}$ assuming $16\,(K_i^{j,l})^2 - 3\,(K_i^{j,u})^2 > 0$, then bound for each $w_i^j(t)$ is

$$|w_i^j(t)| \le 2\,K_i^{j,u} A_i\,\frac{8\,(K_i^{j,l})^2 - (K_i^{j,u})^2 + 12\,K_i^{j,l}K_i^{j,u}}{K_i^{j,l}(16\,(K_i^{j,l})^2 - 3\,(K_i^{j,u})^2} = A_i C_{ij}$$

If we use Euclidean norm for the vector norm in R^n, we get the formula for computation of the local error of $O(h^3)$, which is

$$\begin{aligned}&(1 - Lh/2)(x(t_{k+1}) - y(t_{k+1})) \le\\ &\quad\frac{h^3}{24}H(K+BA)\sum_{i=1}^{n}\sum_{j=1}^{n} K_i^{j,u}A_j(1+C_{ij})\sqrt{n}\\ &\quad+\frac{h^2}{4}\left(HK + H\sqrt{\sum_{i=1}^{n}\left(\sum_{j=1}^{n} K_i^{j,u}C_{ij}\right)^2} + L^2\right)\\ &\quad\times\sqrt{\sum_{i=1}^{n}\left(\sum_{j=1}^{n} K_i^{j,u}A_j(1+C_{ij})\right)^2}\;\frac{e^{\Lambda h}-1}{\Lambda}.\end{aligned} \tag{8}$$

There is a trade off in using the approximation given in III-B.2 vs. the ones given in III-B.1. A comparison of the methods proposed in terms of the error and computational complexity will be presented elsewhere.

The local error at time t_k, consists of two parts. The first part is the analytical error obtained as the difference of the solution of the exact equation (1) and the solution of its approximate equation (2), e.g. the formula given by inequality (8). The second part is the numerical error which is an interval remainder of the Taylor model representing an inclusion of the solution $y(t_k)$ (see Definition 6). We represent the solution set $S_k = \{h(s) \pm \varepsilon \,|\, s \in [-1,+1]\}$, at time t_k, as a Taylor model whose interval remainder consists of both numerical and analytical error. This guarantees the inclusion $S(t_k) \subseteq S_k$, where $S(t_k)$ is the solution set of (1), at time t_k. Thus, S_k is the desired over-approximation of the solution set $S(t_k)$.

The reason we seek local error of $O(h^3)$ is so that if we were to compute the global error (cumulative error for the time of computation, $[t_0, T]$), then we can expect it to be roughly of $O(h^2)$. Our method only guarantees a local error of $O(h^3)$ at the sequence of rational points $\{t_k\}$ which is appriori chosen. However, if one is trying to estimate the error at times $t_k < t < t_{k+1}$ for any k, a different formula should be used, e.g. a logarithmic norm estimate Theorem 5. In this case we cannot guarantee local error of third order.

IV. DERIVATION OF ERROR ESTIMATES

In this section we derive $O(h^3)$ local error bound in the one-dimensionional case with additive noise, i.e. we assume that $n = 1$, $m = 1$, and $g(\cdot) = 1$, $|v(t)| \le A$ (in general, if $g(\cdot) = c$, we can set $g(\cdot) = 1$, and $|v(t)| \le A\,c$).

The input-affine system and its approximation are

$$\dot{x}(t) = f(x(t)) + v(t),\ x(t_k) = x_k; \tag{9}$$

$$\dot{y}(t) = f(y(t)) + w(t),\ y(t_k) = y_k. \tag{10}$$

If $w = 0$, the local error is $|x(t_{k+1}) - y(t_{k+1})| \le 2A\,\frac{e^{\Lambda h}-1}{\Lambda}$. If $w = \int_{t_k}^{t_{k+1}} v(s)ds$, then the local error is $|x(t_{k+1}) - y(t_{k+1})| \le 2LAh\,\frac{e^{\Lambda h}-1}{\Lambda}$. On the other hand, if $w(\cdot)$ is a linear function, which we find by solving the following integral equalities

$$\int_{t_k}^{t_{k+1}} v(t) - w(t)\,dt = 0 \tag{11}$$

$$\int_{t_k}^{t_{k+1}} (v(t) - w(t))(t-(t_k+h/2))\,dt = 0, \tag{12}$$

then we get that $w(t) = a_0 + a_1(t-(t_k+h/2))$ is explicitly given in terms of first and second integrals of $v(\cdot)$ over the time step. In particular, $a_0 = (1/h)\hat{v}$, $a_1 = (12/h^3)\hat{\hat{v}}$, where

$$\hat{v} = \int_{t_k}^{t_{k+1}} v(t)dt; \qquad \hat{\hat{v}} = \int_{t_k}^{t_{k+1}} v(t)(t-(t_k+h/2))\,dt. \tag{13}$$

From above, it is easy to see that $|w(t)| \le 5A/2$. Since we take $x_k = y_k$, the error over time step is computed by the following:

$$\begin{aligned}x(t_{k+1}) - y(t_{k+1}) &= \int_{t_k}^{t_{k+1}} f(x(t)) - f(y(t))dt\\ &\quad+\int_{t_k}^{t_{k+1}} v(t) - w(t)dt\end{aligned}$$

By (11) the second term on the right is zero. Integrating by parts the first term, we get

$$\begin{aligned}x(t_{k+1}) - y(t_{k+1}) &= \left[(t-t_k)\,(f(x(t)) - f(y(t)))\right]_{t_k}^{t_{k+1}}\\ &\quad-\int_{t_k}^{t_{k+1}} (t-t_k)\left(\frac{d}{dt}f(x(t)) - \frac{d}{dt}f(y(t))\right)dt\\ &= h\,(f(x(t_{k+1})) - f(y(t_{k+1})))\\ &\quad-\int_{t_k}^{t_{k+1}} (t-t_k)\left(f'(x(t))\,\dot{x}(t) - f'(y(t))\,\dot{y}(t)\right)dt\end{aligned}$$

Note that we can rewrite the following

$$\begin{aligned}&f'(x(t))\,\dot{x}(t) - f'(y(t))\,\dot{y}(t)\\ &\quad= f'(x(t))\,(\dot{x}(t) - \dot{y}(t)) + (f'(x(t)) - f'(y(t)))\,\dot{y}(t)\\ &\quad= f'(x(t))(f(x(t)) - f(y(t))) + f'(x(t))(v(t) - w(t))\\ &\qquad+(f'(x(t)) - f'(y(t)))\dot{y}(t).\end{aligned}$$

By Mean Value Theorem, there exists $z(\cdot)$ such that $f(x(t)) - f(y(t)) = f'(z(t))(x(t)-y(t))$, and therefore, we get

$$x(t_{k+1}) - y(t_{k+1}) =$$
$$h f'(z(t_{k+1}))\,(x(t_{k+1}) - y(t_{k+1})) \quad (14)$$
$$- \int_{t_k}^{t_{k+1}} (t-t_k)\, f'(x(t))\,(f(x(t)) - f(y(t)))\, dt \quad (15)$$
$$- \int_{t_k}^{t_{k+1}} (t-t_k)\, f'(x(t))\,(v(t) - w(t))\, dt \quad (16)$$
$$- \int_{t_k}^{t_{k+1}} (t-t_k)\,\big(f'(x(t)) - f'(y(t))\big)\,\dot{y}(t)\, dt. \quad (17)$$

Denote by $[0,T]$ an interval of existence for (9) and (10). Let $D \subset \mathbb{R}^n$ be a compact region, such that if $z(t)$ is a solution to (9) or (10) with $z(0) = z_0 \in D$ then $|f(z(t))| \le K$ for all $z \in D$ and all $t \in [0,T]$. Since f is a C^2 function, then first and second order derivatives are uniformly bounded on D. Therefore, there exist Lipschitz constants L and H such that $|f'(\cdot)| \le L$ and $|f''(\cdot)| \le H$. Notice that

$$|\dot{x}(t)| \le K + A, \text{ and } |\dot{y}(t)| \le K + 5A/2.$$

On the other hand, it is useful to see that $|\dot{x}(t) - \dot{y}(t)| \le \Lambda |x(t) - y(t)| + 7A/2$ where Λ is the bound on the logarithmic norm of f', i.e., $\lambda(f'(\cdot)) \le \Lambda$. The above inequality follows from (5). Solving this differential inequality we get $|x(t) - y(t)| \le \frac{7A}{2}\frac{e^{\Lambda h}-1}{\Lambda}$ for $t \in [t_k, t_{k+1}]$. Then, by the integral inequality, and again, by the mean value theorem we get bounds of the absolute values of each of the above

$$|(15)| \le \frac{h^2}{2} \sup_{t\in[t_k,t_{k+1}]} |f'(x(t))|\,|f'(z(t))|\,|x(t) - y(t)|$$

$$|(16)| = \Big| -\Big[f'(x(t)) \int_{t_k}^{t} (t-t_k)(v(t) - w(t))dt\Big]_{t_k}^{t_{k+1}} + \int_{t_k}^{t_{k+1}} f''(x(t))\dot{x}(t) \int_{t_k}^{t} (s-t_k)(v(s) - w(s))\, ds\, dt \Big|$$
$$\le \frac{h^3}{6} \sup_{t\in[t_k,t_{k+1}]} |f''(x(t))\dot{x}(t)|\ |v(t) - w(t)|$$

$$|(17)| \le \frac{h^2}{2} \sup_{t\in[t_k,t_{k+1}]} |f''(z(t))|\ |x(t) - y(t)|\ |\dot{y}(t)|$$

If we combine the above terms, we get the local error is

$$(1 - hL)\,|x(t_{k+1}) - y(t_{k+1})| \le \frac{7}{12} h^3 A H (K + A) + \frac{7}{4} h^2 A \left(L^2 + H(K + 5A/2)\right) \frac{e^{\Lambda h} - 1}{\Lambda} \quad (18)$$

Remark 7: At each time step, we obtain high order estimate on the error, but for any $t \in (t_k, t_{k+1})$ the error might be larger. In particular, we might have to use different formula to find its bound. However, this approach is useful because it will prevent rapid growth of the global error which will be of $O(h^2)$.

Remark 8: For more accurate results, error estimates in (15) and (17) could be increased to $O(h^4)$ with no additional assumptions on the smoothness of the function f. This can be obtained by integrating by parts once more. However, to obtain an error of $O(h^4)$ for the whole equation, f should be C^3, and $w(\cdot)$ should be quadratic.

V. ALGORITHM

In this section we present an algorithm for computation of the solution set of (1), using the single step computation presented earlier.

Algorithm 9: Let $S_k = \{h_k(s) \pm e_k \mid s \in [-1,+1]^{p_k}\}$ be an over-approximation of the set $S(x_0, t_k)$. To compute an over-approximation S_{k+1} of $S(x_0, t_{k+1})$:

1) Compute the flow $\tilde{\phi}_k(x_k, a_k)$ of
$$\dot{x}(t) = f(x(t)) + \textstyle\sum_{i=1}^{m} g_i(x(t)) w(a_{k,i}, t)$$
for $t \in [t_k, t_{k+1}]$, $x_k = x(t_k) \in S_k$, and $a_k \in A$.
2) Add the uniform error bound ε (inequality (8)).
3) Compute the set S_{k+1} which approximates $S(x_0, t_{k+1})$.
4) Reduce the number of parameters (if necessary).
5) Split the new obtained domain (if necessary).

Step (1) of the algorithm produces an approximated flow in the form $\tilde{\phi}_k(x_k, a_k) \approx \phi(x_k, w(a_k, \cdot))$. which is guaranteed to be valid for all $x_k \in S_k$. In practice, we cannot represent $\tilde{\phi}$ exactly, and instead use a polynomial approximation with guaranteed error bound $\hat{\phi}$. Such an approximation is known as a Taylor model. The calculus of Taylor models was first developed by Berz and Makino; see [12] and references therein.

In step (2), we compute the uniform error bound ε_k to make sure an over-approximation is achieved. In step (3), we compute a new approximating set by applying the approximated flow to the initial set of points to obtain a solution set S_{k+1}. Steps (4) and (5) are crucial for the efficiency of the algorithm. It is important to notice that the number of parameters is growing over the time steps. At each time-step, we need an extra $2m$ parameters to represent the effect of the input. This will have a huge impact on the performance of the algorithm.

The easiest way to reduce the number of parameters is to replace the parameter dependency by a uniform error, but this can have a negative impact on accuracy. Another way to reduce number of parameters is using orthogonalization, though this is only possible for affine approximations using currently known methods. If the approximating set becomes too large, it may be hard to compute good approximations to the flow and/or the error. In this case, we can split the set into smaller pieces, and evolve each piece separately. This can improve the error, but is of exponential complexity in the state-space dimension.

VI. NUMERICAL RESULTS

A. Perturbed Van Der Pol Oscillator

We consider the perturbed Van der Pol oscillator given by

$$\dot{x} = y$$
$$\dot{y} = -x + 2(1 - x^2)y + v,$$

where v represents bounded noise. In terms of our general set up (Section II-B), we have $g_i = 1$, for $i = 1, 2$. Therefore,

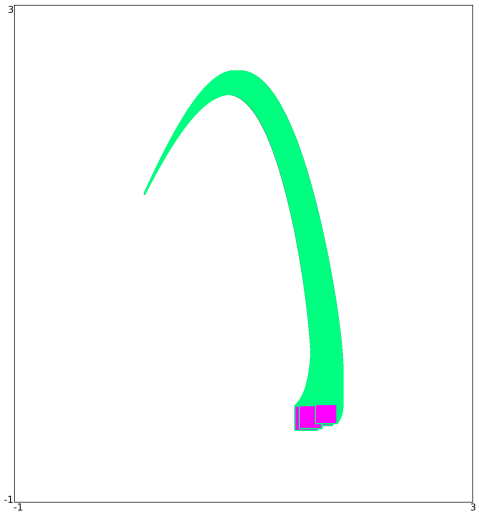

Fig. 1. Evolution of the Perturbed Van Der Pol Oscillator on the state space $X = [-1,3] \times [-1,3]$, with initial points $X_0 = [0.1, 0.105] \times [1.5, 1.505]$, noise $v(\cdot) \in [-0.08, 0.08]$, and time step $h = 0.001$.

suitable formula for computation of the local error for the perturbed Van der Pol oscillator is given by inequality (18), where the absolute value denotes the vector norm in R^2. We take the sup-norm for matrix and vector norms. If we take $D = [0,2] \times [-1,3]$ to be the region of computation, then we get $K = 20$, $L = 31$, $\Lambda = 27$, and $H = 12$. In addition, if we assume that $v(\cdot) \in [-0.08, 0.08]$, we get

$$\varepsilon = ||x(t_{k+1}) - y(t_{k+1})|| \leq 11.24\,h^3 + 168.17\,h^2\,\frac{e^{27h} - 1}{27}$$

We use the algorithm described in Section V to compute the solution set of the perturbed Van der Pol oscillator for the set of initial points $X_0 = [0.1, 0.105] \times [1.5, 1.505]$ over the time interval $[0,3]$. Because the bounds K, L, Λ, and H are rather large, we use fairly small step size, $h = 0.001$, yielding a single-step error of $\varepsilon = 1.817092608 * 10^{-7}$. In Fig. 1, we show the results of the computation of the solution set of the perturbed Van der Pol oscillator using the above values. Splitting of the domain (in half) was performed at $t_1 = 1.2$, and $t_2 = 2.4$.

Fig. 1 shows that our method can be used for rigorous over-approximation of the solution set of a differential inclusion.

VII. CONCLUSIONS AND FUTURE WORK

A. Conclusions

An algorithm for computation of rigorous over-approximations to the set of reachable points of input-affine system with noisy input was presented. The construction gave third-order error estimates at each time step, which is an improvement over the first-order errors previously available in the literature. We presented computation of the error in one-dimensional case and gave a formula for computation in higher-dimensions. In addition, we included a computation of the solution set for the perturbed harmonic oscillator in order to show that efficient and good numerical solutions can be obtained via proposed algorithm.

B. Future Work

It is in our interest to further investigate which method "simple equations" or "full approximation," is better to use in terms of computational efficiency. The algorithm has to be implemented and further tested using the Taylor model functionality available in Ariadne. In addition, reduction of parameters of the algorithm has to be investigated for efficiency purposes. Extensions to even higher order should be possible. Further, we will look for possible extensions of the algorithm to more general differential inclusions, such as those which are nonlinear in the input.

VIII. ACKNOWLEDGEMENTS

This research was partially supported by the European Commission through the project "Control for Coordination of Distributed Systems" (C4C) as part of the EU.ICT program (challenge ICT-2007.3.7).

REFERENCES

[1] Ariadne. http://trac.parades.rm.cnr.it/ariadne/
[2] Aubin, J.-P., and C., Arrigo, Differential inclusions. Set-valued maps and viability theory. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 264. Springer-Verlag, Berlin, 1984.
[3] Beyn, W.-J., and Rieger, J., Numerical fixed grid methods for differential inclusions. (English summary) *Computing.* 81 (2007), no. 1, 91-106.
[4] Collins, P., and Graca, D. S. Effective computability of solutions of differential inclusions: the ten thousand monkeys approach. *J.UCS* 15 (2009), no. 6, 1162-1185.
[5] Deimling, K., *Multivalued differential equations.* De Gruyter Series in Nonlinear Analysis and Applications, 1. Walter de Gruyter and Co., Berlin, 1992.
[6] Dontchev, A. L.; Farkhi, E. M. Error estimates for discretized differential inclusion. *Computing.* 41 (1989), no. 4, 349-358.
[7] Filippov, A. F. *Differential equations with discontinuous righthand sides.* Translated from the Russian. Mathematics and its Applications (Soviet Series), 18. Kluwer Academic Publishers Group, Dordrecht, 1988.
[8] Grammel, G., Towards fully discretized differential inclusions. *Set-Valued Anal.* 11 (2003), no. 1, 1-8.
[9] Hairer, E.; Nrsett, S. P.; Wanner, G. *Solving ordinary differential equations. I. Nonstiff problems.* Springer Series in Computational Mathematics, 8. Springer-Verlag, Berlin, 1987
[10] Kurzhanski, A.; Valyi, I., *Ellipsoidal calculus for estimation and control.* (English summary) Systems and Control: Foundations and Applications. Birkhuser Boston, Inc., Boston, MA; International Institute for Applied Systems Analysis, Laxenburg, 1997.
[11] Puri, A.; Borkar, V. and Varaiya, P., ε-approximation of differential inclusions, *Proc. of the 34th IEEE Conference on Decision and Control* (1995), pp. 28922897.
[12] Revol, N.; Makino, K.; Berz, M. Taylor models and floating-point arithmetic: proof that arithmetic operations are validated in COSY. *J. Log. Algebr. Program.* 64 (2005), no. 1, 135–154.
[13] Smirnov, G. V., *Introduction to the theory of differential inclusions.* Graduate Studies in Mathematics, 41. American Mathematical Society, Providence, RI, 2002.
[14] Wade, R. W., *An Introduction to Analysis.* Pearson Prentice Hall, Upper Saddle River, NJ, 2009.
[15] Weihrauch, K., *Computable analysis. An introduction.* Texts in Theoretical Computer Science. An EATCS Series. Springer-Verlag, Berlin, 2000.
[16] Zgliczynski, P.; Kapela, T. A., Lohner algorithm for perturbation of ODEs and differential inclusions. *Discrete Contin. Dyn. Syst. Ser. B* 11 (2009), no. 2, 365–385.